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Topological contributions to the partition function for 2D massive fermions

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Abstract. We study the contribution of topologically non-trivial sectors to the vacuum functional Z_m of a model of massive fermions in two dimensions. By using recent results on non-minimal correlation functions we can write Z_m as a kind of sector expansion. In this series, the contribution of each sector is given by integrals of minimal correlation functions evaluated in the massless theory. Finally we comment on the application of our result to the analysis of some special cases. In particular we briefly show that Coleman's equivalence between the massive Thirring and sine-Gordon models is not affected when topology is taken into account.

Following the inspiring paper by Bardakci and Crescimanno [1], a systematic procedure for the analysis of fermionic 2D models in topological backgrounds has been developed [2–4]. In particular, the minimal correlation functions first defined in [1], have been computed for Abelian (Thirring, Schwinger) [2, 3] and non-Abelian (QCD_2) [4] models.

In a more recent paper [5] the more complex non-minimal correlation functions [1] have been explicitly evaluated. It has been stressed in [5] that these functions would play a crucial role when considering massive fermions. In fact, in previous investigations, only massless fermions were taken into account. In this condition it has been established, as a general rule, that the partition function $Z=\Sigma_N Z_N$ (N indicates the topological sector) receives the contribution of the $N=0$ sector only. This raises an interesting question about the role that the fermion mass will play when computing the partition function. In other words: will non-trivial sectors contribute to the massive partition function Z_m?

The purpose of this work is to answer this question. To this end we have applied the path-integral method of [5] to the model considered in [1] with massive fermions. We have found that all topological sectors contribute to Z_m and we have obtained a closed expression for each contribution in terms of the minimal functions corresponding to the massless theory.

We start from the partition function

$$Z_m=\sum_N \int \mathrm{D}A_\mu^N\,\mathrm{D}\bar\Psi\,\mathrm{D}\Psi\,\exp\left[-\int \mathrm{d}^2x\,\bar\Psi(\mathrm{i}\partial\!\!\!/+m+A\!\!\!/^N)\Psi\right] \qquad (1)$$

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where $\bar{\Psi}$, Ψ are massive Dirac fermion fields coupled to an Abelian vector field which carries a topological charge N. The partition function Z_m is then written as a sum over all topological sectors.

Note that the massive Dirac operator in (1) does not have zero modes solutions [6]. However, one can find a relation with the massless case by performing a perturbative expansion in the mass. Indeed, writing the mass term as

$$m\bar{\Psi}(x)\Psi(x)=m[S_+(x)+S_-(s)] \tag{2}$$

with

$$S_+=\bar{\Psi}_R\Psi_R \qquad S_-=\bar{\Psi}_L\Psi_L$$

and

$$\Psi=\begin{pmatrix}\Psi_R\\ \Psi_L\end{pmatrix} \qquad \bar{\Psi}=(\bar{\Psi}_R\bar{\Psi}_L)$$

we obtain

$$Z_m=Z_0\left\langle \exp\left(-\int d^2x\, m(S_++S_-)\right)\right\rangle_0 \tag{3}$$

where $\langle\ \rangle_0$ means v.e.v. with respect to the massless theory. The expansion in the mass can then be written as:

$$Z_m=Z_0\sum_{j=0}^{\infty}\frac{m^j(-1)^j}{j!}\mathscr{Z}_j \tag{4}$$

where $\mathscr{Z}_0=1$, and to $j\geqslant 1$:

$$\mathscr{Z}_j=\left\langle\int\left(\prod_{i=1}^{j} d^2x_i\right)\prod_{i=1}^{j}(S_++S_-)(x_i)\right\rangle_0. \tag{5}$$

At this stage it becomes apparent that $\mathscr{Z}_j$ will be a sum of minimal and non-minimal functions [5]. To illustrate this point, let us write explicitly $\mathscr{Z}_3$:

$$\begin{aligned}\mathscr{Z}_3&=\left\langle\int d^2x_0\, d^2x_1\, d^2x_2(S_++S_-)(x_0)(S_++S_-)(x_1)(S_++S_-)(x_2)\right\rangle_0\\ &=\int d^2x_0\, d^2x_1\, d^2x_2[\langle S_+(x_0)S_+(x_1)S_+(x_2)\rangle+\langle S_-(x_0)S_-(x_1)S_-(x_2)\rangle\\ &\quad+3\langle S_+(x_0)S_+(x_1)S_-(x_2)\rangle+3\langle S_-(x_0)S_-(x_1)S_+(x_2)\rangle]\\ &=\int d^2x_0\, d^2x_1\, d^2x_2[\mathscr{M}_{(3)}(x_0,x_1,x_2)+\mathscr{M}_{(-3)}(x_0,x_1,x_2)\\ &\quad+3(\mathscr{N}^3_{(1)}(x_0,x_1,x_2)+\mathscr{N}^3_{(-1)}(x_0,x_1,x_2)].\end{aligned} \tag{6}$$

In the above expression we have used the general definitions [1, 5]:

$$\mathcal{M}_{(\pm|N|)}(x_0, \ldots, x_{|N|-1}) = \langle S_\pm(x_0) \ldots S_\pm(x_{|N|-1})\rangle_0 \tag{7}$$

$$\begin{aligned}\mathcal{N}^n_{(\pm|N|)}(x_0, \ldots, x_{n-1}) \\ &= \langle S_\pm(x_0)S_\pm(x_1) \ldots S_\pm(x_{|N|-1})S_\pm(x_{|N|}) \ldots S_\pm(x_{|N|+\frac{1}{2}(n-|N|)-1}) \\ &\quad S_\mp(x_{\frac{1}{2}(|N|+n)}) \ldots S_\mp(x_{n-1})\rangle_0\end{aligned} \tag{8}$$

where the subscript $(\pm|N|)$ denotes the topological sector that contributes to each function. For minimal functions $\mathcal{M}_{\pm|N|}$, $|N|$ has been shown to be equal to the number of points [1, 2], whereas for the non-minimal correlations $\mathcal{N}^n_{\pm N}$, the number of points n is greater than $|N|$.

Now it is a simple exercise to write the jth contribution to Z_m as:

$$\mathscr{Z}_j = \int \prod_{i=0}^{j-1} d^2x_i \left[\mu_j(x_0, \ldots, x_{j-1}) + \sum_{k=1}^{j-1} \binom{j}{j-k} \mathcal{N}^j_{(j-2k)}\right] \tag{9}$$

for $j \geqslant 2$ and

$$\mathscr{Z}_1 = \int d^2x\, \mu_1(x) \tag{10}$$

with

$$\mu_j(x_0, \ldots, x_{j-1}) = \mathcal{M}_j(x_0, \ldots, x_{j-1}) + \mathcal{M}_{-j}(x_0, \ldots, x_{j-1}). \tag{11}$$

Due to the presence of the integrals in the expansion of Z_m, by conveniently renaming coordinates, one can relate the non-minimal function $\mathcal{N}^j_{j-2k}$ appearing in (9) with the one obtained in [5] as follows:

$$\begin{aligned}\int d^2x_0 \ldots d^2x_{j-1}\mathcal{N}^j_{(j-2k)} \\ &= \int d^2x_0 \ldots d^2x_{j-1} \binom{\frac{1}{2}(j+|j-2k|)}{\frac{1}{2}(j-|j-2k|)} \mathcal{N}^j_{1,(j-2k)}(x_0, \ldots, x_{j-1}).\end{aligned} \tag{12}$$

From now on we will drop the first subscript 1 in the RHS of (12).

Replacing (12) in (9) we obtain:

$$\begin{aligned}\mathscr{Z}_j = \int \prod_{i=0}^{j-1} d^2x_i \Bigg[\mu_j(x_0, \ldots, x_{j-1}) \\ &+ \sum_{k=1}^{j-1} \binom{j}{j-k} \binom{\frac{1}{2}(j+|j-2k|)}{\frac{1}{2}(j-|j-2k|)} \mathcal{N}^j_{(j-2k)}(x_0, \ldots, x_{j-1})\Bigg].\end{aligned} \tag{13}$$

In [5] we have shown that there exists a particular distribution for the topological charge for which the non-minimal function $\mathcal{N}^j_{(j-2k)}(x_0, \ldots, x_{j-1})$, that appears in (13), can be factorized in terms of minimal functions $\mathcal{M}_{(j-2k)}(x_0, \ldots, x_{|j-2k|-1})$ and fermionic Green functions. The minimal correlations take into account the contribution of the zero modes of the model.

We obtain for the $\mathcal{N}^j_{(j-2k)}(x_0, \ldots, x_{j-1})$ functions the following form:

$$\mathcal{N}^j_{(j-2k)}(x_0, \ldots, x_{j-1}) = \frac{\mathcal{M}_{(j-2k)}(x_0, \ldots, x_{|j-2k|-1})}{\det_0(i\not\partial + \not A^c)} \times \tilde{G}^{(0)}_{j-|j-2k|}(x_{|j-2k|}, \ldots, x_{j-1}) \tag{14}$$

where $\det_0(i\not\partial + \not A^c)$ is the determinant without zero modes with $\not A^c$ carrying a charge $j-2k$, and

$$\tilde{G}^{(0)}_{2k}(x_0, \ldots, x_{2k}) = G^{(0)}_{2k}(x_0, \ldots, x_{2k}) \times F(x_0, \ldots, x_{2k}) \tag{15}$$

where $G^{(0)}_{2k}$ is a $2k$-point amplitude for massless fermions ($N=0$ sector) and F is a certain function which depends on the same coordinates as $G^{(0)}_{2k}$. The explicit form of F is rather involved and we do not reproduce it here. The interested reader can find it in [5]. The relevant point here is that both factors in (14) ($\mathcal{M}$ and $\tilde{G}$) are functions of a different set of coordinates. This coordinate decoupling will enable us to rewrite (4) in a more illuminating way. Indeed, by carefully analysing the contribution of every topological sector to the term corresponding to each order in the mass expansion, we shall be able to write Z_m in the form:

$$Z_m = \sum_N Z_m^{(N)}. \tag{16}$$

As all sectors contribute to one given order m^n, that is not, in principle, an easy task. However (14) will greatly simplify this problem by allowing us to formally sum the contributions of the non-topological $\tilde{G}^{(0)}$ factors. Inserting (15) in (13) we get:

$$\mathscr{Z}_j = \int \prod_{i=0}^{j-1} d^2x_i \Bigg[\mu_j(x_0, \ldots, x_{j-1}) + \sum_{k=1}^{j-1} \binom{j}{j-k} \binom{\frac{1}{2}(j+|j-2k|)}{\frac{1}{2}(j-|j-2k|)} \times \frac{\mathcal{M}_{(j-2k)}(x_0, \ldots, x_{|j-2k|-1})}{\det_0(i\not\partial + \not A^c)} \times \tilde{G}^{(0)}_{j-|j-2k|}(x_{|j-2k|}, \ldots, x_{j-1}) \Bigg]. \tag{17}$$

Now we can replace (17) in (4) to obtain:

$$Z_m = Z_0 \left[1 + \Gamma(m) + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} m^i \int d^2x_0 \ldots d^2x_{i-1} \mu_i(x_0, \ldots, x_{i-1}) \left(1 + \frac{\hat{\mu}_i}{\mu_i} \tilde{\Gamma}(m) \right) \right] \tag{18}$$

where

$$\Gamma(m) = \sum_{k=1}^{\infty} \frac{m^{2k}}{(k!)^2} \int d^2x_0 \ldots d^2x_{2k-1} G^{(0)}_{2k}(x_0, \ldots, x_{2k-1}) \tag{19}$$

and $\tilde{\Gamma}(m)$ is given by (19) with $G^{(0)}_{2k}$ replaced by $\tilde{G}^{(0)}_{2k}$ (see equation (15)). We have also defined:

$$\hat{\mu}_i(x_0, \ldots, x_{i-1}) = \frac{\mathcal{M}_i(x_0, \ldots, x_{i-1})}{\det_0^{(i)}(i\not\partial + \not A^c)} + \frac{\mathcal{M}_{-i}(x_0, \ldots, x_{i-1})}{\det_0^{(-i)}(i\not\partial + \not A^c)} \tag{20}$$

where $\det_0^{(i)}(i\not\partial + \not A^c)$ is the fermionic determinant without zero modes and with $\not A^c$ carrying a charge i.

Now we can use the explicit form for $\mathcal{M}_i$ in order to show that $\hat{\mu}_i/\mu_i$ does not depend on the coordinates ([5]):

$$\mathcal{M}_i = \det{}_0^{(i)}(\mathrm{i}\partial\!\!\!/ + A\!\!\!/^{c}) \prod_{j=0}^{|i|-1} |x_i - x_j|^{-2}. \tag{21}$$

We then have

$$\frac{\hat{\mu}_i}{\mu_i} = \frac{2}{\det{}_0^{(i)}(\mathrm{i}\partial\!\!\!/ + A\!\!\!/^{c}) + \det{}_0^{(-i)}(\mathrm{i}\partial\!\!\!/ + A\!\!\!/^{c})} \tag{22}$$

and thus the massive partition function can be finally cast in the form:

$$Z_m = Z_0 \left[1 + \Gamma(m) + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} m^i \left(1 + \frac{2\tilde{\Gamma}(m)}{\det^{(i)} + \det^{(-i)}} \right) \right.$$
$$\left. \times \int \mathrm{d}^2x_0, \ldots, \mathrm{d}^2x_{i-1} \mu_i(x_0, \ldots, x_{i-1}) \right]. \tag{23}$$

This is our main result. We have been able to write Z_m as a sum of contributions, each coming from one particular topological sector. This is in contrast with all previous calculations for the massless case where Z is not modified by the presence of the topological background. The price we have paid in order to get this sector expansion is that now the dependence on the mass has become implicit through $\Gamma(m)$ and $\tilde{\Gamma}(m)$. On the other hand the dependence on the topological sector is explicit now in terms of integrals of minimal correlations. Note that $\Gamma(m)$, $\tilde{\Gamma}(m) \to 0$ when $m \to 0$, and we reobtain the massless result, as expected.

We want to stress here that although (23) was obtained for the model considered in [1], the same procedure could be followed for other 2D models in order to get a similar expansion. In particular for the massless Thirring model it has been shown that all minimal functions vanish for any finite value of the couplnig constant [2]. This means that for the massive Thirring model we shall have

$$Z_m^{\text{Thirring}} = Z_{m=0}[1 + \Gamma(m)] = Z_m^{\text{Thirring}, N=0}. \tag{24}$$

Therefore we see that for this particular model only the $N=0$ sector contributes to the partition function, even in the massive case. This fact raises an interesting question concerning the validity of the bosonization rules, relating the massive Thirring and sine-Gordon models [7], when topology is taken into account.

Let us consider the sine-Gordon vacuum functional given by

$$Z_{\mathrm{SG}} = \sum_N \int \mathrm{D}\phi_N \exp\left\{ -\int \mathrm{d}^2x \left[\frac{1}{2} (\partial_\mu \phi_N)^2 - \alpha \cos(\beta \phi_N) \right] \right\} \tag{25}$$

where topology is included by writing

$$\phi_N = \varphi_N^{\mathrm{cl}}(x) + \varphi(x) \tag{26}$$

such that $\mathrm{D}\phi_N = \mathrm{D}\varphi$ and $\int \partial_\mu \partial_\mu \varphi_N^{\mathrm{cl}} = 2\pi N$.

Performing an expansion in α and evaluating the resulting v.e.v.'s with respect to the free bosonic model we obtain:

$$Z_{\mathrm{SG}}=KZ_{\mathrm{SG}}^{N=0} \tag{27}$$

where $K=\Sigma_N$ is an infinite constant which can be absorbed as a normalization (it will disappear when computing v.e.v.'s).

We then conclude that Coleman's equivalence between the massive Thirring and sine-Gordon models will hold when non-trivial topological sectors are taken into account.

A similar analysis can be done for QED_2 with massive fermions (massive Schwinger model [8]). In this case the minimal correlations are non-vanishing and thus a non-trivial change will take place in Z_m [3]. As in the Thirring case, the equivalence between this model and a massive sine-Gordon can be explored in this context.

Concerning the sine-Gordon action, the only change is the addition of a mass term $\frac{1}{2}\mu^2\phi_N^2$. One can then follow the same steps that led us to (27). Thus, one obtains

$$Z_{\mathrm{SG},\mu}=KZ_{\mathrm{SG},\mu}^{N=0}. \tag{28}$$

The Lagrangian density for massive QED_2 is given by

$$\bar{\Psi}(\mathrm{i}\partial\!\!\!/+m+eA\!\!\!/)\Psi+\tfrac{1}{4}F_{\mu\nu}^2. \tag{29}$$

Specializing (4) for the corresponding vacuum functional we get

$$Z_{\mathrm{Schw},m}=Z_{\mathrm{Schw},0}\sum_{j=0}^{\infty}\frac{(-m)^j}{j!}\mathscr{Z}_j \tag{30}$$

with $\mathscr{Z}_j$ given by (9). For our present purpose it is convenient to rewrite $\mathscr{Z}_j$ as

$$\mathscr{Z}_j=\int\prod_{i=0}^{j-1}\mathrm{d}^2x_i\sum_{k=0}^{j-1}\binom{j}{j-k}\mathcal{N}_{(j-2k)}^j(x_0,\ldots,x_{j-1}) \tag{31}$$

where we have renamed μ_j in the form

$$\mu_j=\mathcal{N}_j^j. \tag{32}$$

At this point we can extract the $N=0$ contribution from (31) by isolating the term that corresponds to $k=j/2$:

$$\mathscr{Z}_j=\int\prod_{i=0}^{j-1}\mathrm{d}^2x_i\frac{j!}{(j!/2)^2}\mathcal{N}_0^j+\int\prod_{i=0}^{j-1}\mathrm{d}^2x_i\sum_{\substack{k=0\\k\neq j/2}}^{j-1}\binom{j}{j-k}\mathcal{N}_{(j-2k)}^j. \tag{33}$$

One can verify that the first term in the right-hand side of (33) when replaced in (30) allows us to obtain

$$Z_{\mathrm{Schw},m}=Z_{\mathrm{Schw},m}^{N=0}+Z_{\mathrm{Schw},0}^{N=0}\sum_{j=0}^{\infty}\frac{(-m)^j}{j!}\int\prod_{i=0}^{j-1}\mathrm{d}^2x_i\sum_{\substack{k=0\\k\neq j/2}}^{j-1}\binom{j}{j-k}\mathcal{N}_{(j-2k)}^j. \tag{34}$$

Now we use the well known result for $N=0$ [7, 8]:

$$Z_{\mathrm{Schw},m}^{N=0}=Z_{\mathrm{SG},\mu}^{N=0}. \tag{35}$$

This, together with the fact that minimal and non-minimal functions are non-vanishing for the massless Schwinger model [3], shows that $Z_{\mathrm{Schw},m}$ and $Z_{\mathrm{SG},\mu}$ cannot be trivially identified when a topologically non-trivial configuration is present.

In summary, we have studied the contributions of topologically nontrivial sectors to the vacuum-to-vacuum functional Z_m of 2D massive models. Our main result is given by (23) which provides a sector expansion for Z_m. In this context, we have also discussed how the presence of a classical background affects certain bosonization equivalences, well established for $N=0$. In particular, we found that Coleman's result [7] involving the massive Thirring and massless sine-Gordon models remains valid even for $N\neq 0$ (equations (24) and (27)). On the contrary, the partition functions corresponding to massive Schwinger and massive sine-Gordon models [7, 8] are not equal but differ by a sum of minimal and non-minimal functions that picks up the contribution of every non-trivial sector (equations (34) and (35)).

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